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Short Communication

A NOTE ON EXTREMALLY DISCONNECTED SPACES

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Abstract

In this paper we give several equivalent characterizations of extremally disconnected spaces.

Keyword: Preopen set, Semi-open set, Interior, Closure, Regular-open set, Regular-closed set, AB−set, $\alpha-$ open, Insertion of function.

1 Introduction

Extremally disconnected spaces are important in studying the Stone-Čech compactification of a product space [(19I.2) of 14] as well as, more generally, in the study of the Stone space of any complete Boolean algebra. They also crop up in investigations of the reducibility of mappings of compact spaces [(17p) of 14], and the extremally disconnected compact spaces are precisely the compact-projective spaces [(17Q) of 14].
Also, extremally disconnected spaces play a prominent role in set-theoretical topology. Due to their peculiar properties extremally disconnected spaces provide curcial applications in the theory of Boolean algebra, in axiomatic set theory and in some branches of functional analysis (for example: in C^*-algebra) as well. There are many interesting equivalent of extremally disconnected spaces. See for example: problems [1H, 3N, 6M of 6], [(15G.1), (19 I.1) of 14] and, in case of Boolean algebra, Theorem[2.33 of 2].
Recall that a topological space X is said to be extremally disconnected if the closure of every open set of X is open in X or equivalently if the interior of every closed set of X is closed in X. The term 'extremally disconnected' was introduced by M.H. Stone [12].
We denote by int(A) the interior of A and by cl(A) the closure of A. A subset A of a topological space X is called preopen or locally dense or nearly open if $A \subseteq int(cl(A))$. A set A is called preclosed if its complement is preopen or equivalently if $cl(int(A)) \subseteq A$. The term 'preopen' was used for the first by A.S. Mashhour, M.E. Abd El -Monsef and S.N. El-Deeb[9], while the concept of a 'locally dense' set was introduced by H.H. Corson and E. Michael [3].
A subset A of X is called semi-open if $U \subseteq A \subseteq cl(U)$ for some open set $U \subseteq X$ or equivalently if $A \subseteq cl(int(A))$, whereas it is semi-propen if $U \subseteq A \subseteq cl(U)$ for some preopen set $U \subseteq X$.
A subset A of X is called $\alpha-$open if $A \subseteq int(cl(int(A)))$. We have a subset of X is $\alpha-$open if and only if it is semi -open and preopen.
A subset A of X is called regular-open if $int(cl(A)) = A$. A set A is called regular-closed if $cl(int(A)) = A$.
A subset A of X is called semi-regular if there exists a regular-open set U such that $U \subseteq A \subseteq cl(U)$ [8].
A subset A of X is called an AB−set if $A = U \cap V$, where U is open and V is semi -regular. The collection of all AB−sets in X will be denoted by AB(X) [4].

In this short note we prove some set-theoretical equivalent of extremally disconnected spaces**.**

2 CHARACTERIZATIONS

Remark 2.1. The following equivalent conditions for a topological space (X, τ) are established. All of the implications are known or obvious:

(1) X is extremally disconnected;
(2) cl(A) ∩ cl(B)= cl(A ∩ B) for all open subsets A and B of X [5]**;**
(3) cl(A) ∩ cl(B)= ∅ for all open subsets A and B of X with A ∩ B = ∅ [5], [(15G.1) of 14]**;**
(4) cl(A) ∩ cl(B)= ∅ for all semi-open subsets A and B of X with A ∩ B = ∅ [11]**;**
(5) int(cl(A)) ∩ cl(B)= ∅ for all subsets A and all open subsets B of X with A ∩ B = ∅ [5]**;**
(6) cl(int(cl(A))) ∩ cl(B)= ∅ for all subsets A and all open subsets B of X with A ∩ B = ∅ [5]**;**
(7) int(cl(A)) ∪ int(cl(B)) = int(cl(A ∪ B)) for all open subsets A and B of X [5]**;**
(8) int(A) ∪ int(B)= int(A ∪ B) for all closed subsets A and B of X [5];
(9) A∩ B is semi-open for all semi-open subsets A and B of X [7];
(10) A ∩ B is semi-open for all regular-closed subsets A and B of X [7]**;**
(11) cl(A) ∩ cl(B) is semi-open for all semi-open subsets A and B of X [7];
(12) cl(A) ∩ cl(B) is semi-open for all open subsets A and B of X [7];
(13) bdry(A)= ∅ for all regular-open subsets A of X [7]**;**
(14) Every regular-open subset of X is closed [7];
(15) Every regular-closed subset of X is open [7];
(16) Every regular-open subset of X is regular-closed and every regular-closed subset of X is regular-open [7]**;**
(17) A ∩ B is regular-closed for all regular-closed subsets A and B of X [7];
(18) A ∪ B is regular-open for all regular-open subsets A and B of X [7];
(19) The closure of every semi-open subset of X is open [11];
(20) The closure of every preopen subset of X is preopen [10];
(21) A ∩ B is semi-open for all semi-open subsets A and all semi-preopen subsets B of X [10];
(22) τ = AB(X) [4]**;**
(23) Every AB−set is open [4]**;**
(24) Every two disjoint open subsets in X are completely separated [(15G.1) of 14]**;**
(25) Every open subspace of X is c^*-embedded [(15G.1) of 14]**;**
(26) Every semi-open subspace of X is c^*-embedded [11]**;**
(27) βX is extremally disconnected [(19 I.1) of 14]**;**
(28) Class semi-open subsets of X is the strongest topology over X among those that have the same class of semi-open subsets as X [7]**;**
(29) For any real valued functions f and g on X with $g \le f$, g has property lower semicontinuous and f has property upper semicontinuous, there exists a continuous function h on X such that $g \le h \le f$ i.e. space X has the weak c−insertion property for (lsc, usc) [13]**;**
(30) Every regular-closed subset of X is preopen [12]**;**
(31) Every semi-open subset of X is preopen [12]**;**
(32) The closure of every preopen subset of X is open [12]**;**

Proofs of the following equivalent conditions are not trivial. But as a good exercise details are left to the readers.

Theorem 2.1. The following conditions are equivalent for a space (X, τ):

(a) X is extremally disconnected;
(b) The closure of every regular-open subset of X is open;
(c) Every semi-open subset of X is α−open;
(d) Class semi-open subsets of X is the same class of α−open subsets as X .

Proof. (a) $\Leftrightarrow$ (b) and ((31) of Remark 2.1.) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ ((31) of Remark 2.1.).

Remark 2.2. Above Theorem seems to be an isolated result, but this will be useful to the readers those are interested in generalization of concepts having closed relation with extremally disconnected spaces. For example: in generalization of certain properties of spaces with filters and ∞ −disconnected (extremally disconnected) spaces (cf. [1]).

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